Russell's paradox

From Wikipedia, the free encyclopedia

Part of the foundations of mathematics, **Russell's paradox** (also known as **Russell's antinomy**), discovered by Bertrand Russell in 1901, showed that the naive set theory of Frege leads to a contradiction.

It might be assumed that, for any formal criterion, a set exists whose members are those objects (and only those objects) that satisfy the criterion; but this assumption is disproved by a set containing exactly the sets that are not members of themselves. If such a set qualifies as a member of itself, it would contradict its own definition as *a set containing sets that are not members of themselves*. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox.

In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and Ernst Zermelo's axiomatic set theory, the first consciously constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical ZFC set theory.

Contents

Informal presentation

Let us call a set "normal" if it does not contain itself as a member. For example, take the set of all squares. That set is not itself a square, and therefore is not a member of the set of all squares. So it is "normal". On the other hand, if we take the complementary set of all non-squares, that set is itself not a square and so should be one of its own members. It is "abnormal".

Now we consider the set of all normal sets – let us give it a name: *R* – and ask the question: is *R* a "normal" set? If it is "normal", then it is a member of *R*, since *R* contains all "normal" sets. But if that is the case, then *R* contains itself as a member, and therefore is "abnormal". On the other hand, if *R* is "abnormal", then it is not a member of *R*, since *R* contains only "normal" sets. But if that is the case, then *R* does not contain itself as a member, and therefore is "normal". Clearly, this is a paradox: if we suppose *R* is "normal" we can prove it is "abnormal", and if we suppose *R* is "abnormal" we can prove it is "normal". Hence, *R* is both "normal" *and* "abnormal," which is a contradiction.

Formal derivation

Let *R* be "the set of all sets that do not contain themselves as members". Formally: *A* is an element of *R* if and only if *A* is not an element of *A*. In set-builder notation:

$$R = \{A \mid A \not\in A\}.$$

Nothing in the system of Frege's *Grundgesetze der Arithmetik* rules out *R* being a well-defined set. The problem arises when it is considered whether *R* is an element of itself. If *R* is an element of *R*, then according to the definition *R* is not an element of *R*. If *R* is not an element of *R*, then *R* has to be an element of *R*, again by its very definition. The statements "*R* is an element of *R*" and "*R* is not an element of *R*" cannot both be true, thus the contradiction.

The following fully formal yet elementary derivation of Russell's paradox[1] makes plain that the paradox requires nothing more than first-order logic with the unrestricted use of set abstraction. The proof is given in terms of collections (all sets are collections, but not conversely). It invokes neither set theory axioms nor the law of excluded middle explicitly or tacitly.

Definition. The collection $\{x : \Phi(x)\}$, in which $\Phi(x)$ is any predicate of first-order logic in which x is a free variable, denotes the individual A satisfying $\forall x\,[x \in A \leftrightarrow \Phi(x)]$.

Theorem. The collection $R = \{x : x \notin x\}$ is contradictory.

Proof. Replace $\Phi(x)$ in the definition of collection by $x \notin x$, so that the implicit definition of R becomes $\forall x\,[x \in R \leftrightarrow x \notin x]$. Instantiating x by R then yields the contradiction $R \in R \leftrightarrow R \notin R$. □

Remark. The above definition and theorem are the first theorem and definition in Potter (2004), consistent with the fact that Russell's paradox requires no set theory whatsoever. Incidentally, the force of this argument cannot be evaded by simply proscribing the substitution of $x \notin x$ for $\Phi(x)$. In fact, there are denumerably many formulae $\Phi(x)$ giving rise to the paradox.[2] For some examples, see reciprocation below.

The paradox holds in intuitionistic logic

The preceding shows that the set $R = \{A \mid A \notin A\}$ leads to a contradiction by showing that assuming *R* true and assuming it false both lead to absurdity; the resulting contradiction implicitly assumes the law of excluded middle. Thus it may be tempting to conclude that the paradox is avoided if the law of excluded middle is disallowed, as with intuitionistic logic. However, the paradox can still be generated by means of the intuitionistically valid law of non-contradiction, as follows.

Theorem. The collection $R = \{x : x \notin x\}$ is contradictory even if the background logic is intuitionistic.

Proof. From the definition of *R*, we have that $R \in R \leftrightarrow \neg(R \in R)$. Then $R \in R \rightarrow \neg(R \in R)$ (biconditional elimination). But also $R \in R \rightarrow R \in R$ (the law of identity), so $R \in R \rightarrow (R \in R \land \neg(R \in R))$. But by the law of non-contradiction we know that $\neg(R \in R \land \neg(R \in R))$. By modus tollens we conclude $\neg(R \in R)$.

But since $R \in R \leftrightarrow \neg(R \in R)$, we also have that $\neg(R \in R) \rightarrow R \in R$, and so we also conclude $R \in R$ by modus ponens. Hence we have deduced both $R \in R$ and its negation using only intuitionistically valid methods. □

More simply, it is intuitionistically impossible for a proposition to be equivalent to its negation. Assume $P \leftrightarrow \neg P$. Then $P \rightarrow \neg P$. Hence $\neg P$. Symmetrically, we can derive $\neg\neg P$, using $\neg P \rightarrow P$. So we have inferred both $\neg P$ and its negation from our assumption, with no use of excluded middle.

Reciprocation

Russell's paradox arises from the supposition that one can meaningfully define a class in terms of any well-defined property $\Phi(x)$; that is, that we can form the set $P = \{x \mid \Phi(x) \text{ is true }\}$. When we take $\Phi(x) = x \notin x$, we get Russell's paradox. This is only the simplest of many possible variations of this theme.

For example, if one takes $\Phi(x) = \neg(\exists z : x \in z \land z \in x)$, one gets a similar paradox; there is no set *P* of all *x* with this property. For convenience, let us agree to call a set *S* *reciprocated* if there is a set *T* with $S \in T \land T \in S$; then *P*, the set of all non-reciprocated sets, does not exist. If $P \in P$, we would immediately have a contradiction, since *P* is reciprocated (by itself) and so should not belong to *P*. But if $P \notin P$, then *P* is reciprocated by some set *Q*, so that we have $P \in Q \land Q \in P$, and then *Q* is also a reciprocated set, and so $Q \notin P$, another contradiction.

Any of the variations of Russell's paradox described above can be reformulated to use this new paradoxical property. For example, the reformulation of the Grelling paradox is as follows. Let us agree to call an adjective *P* "nonreciprocated" if and only if there is no adjective *Q* such that both *P* describes *Q* and *Q* describes *P*. Then one obtains a paradox when one asks if the adjective "nonreciprocated" is itself nonreciprocated.

This can also be extended to longer chains of mutual inclusion. We may call sets $A_1, A_2, \ldots, A_n$ a chain of set A_1 if $A_{i+1} \in A_i$ for $i=1,2,\ldots,n-1$. A chain can be infinite (in which case each A_i has an infinite chain). Then we take the set *P* of all sets which have no infinite chain, from which it follows that *P* itself has no infinite chain. But then $P \in P$, so in fact *P* has the infinite chain *P*,*P*,*P*,... which is a contradiction. This is known as Mirimanoff's paradox.

Set-theoretic responses

In 1908, Ernst Zermelo proposed an axiomatization of set theory that avoided the paradoxes of naive set theory by replacing arbitrary set comprehension with weaker existence axioms, such as his axiom of separation (*Aussonderung*). Modifications to this axiomatic theory proposed in the 1920s by Abraham Fraenkel, Thoralf Skolem, and by Zermelo himself resulted in the axiomatic set theory called ZFC. This theory became widely accepted once Zermelo's axiom of choice ceased to be controversial, and ZFC has remained the canonical axiomatic set theory down to the present day.

ZFC does not assume that, for every property, there is a set of all things satisfying that property. Rather, it asserts that given any set *X*, any subset of *X* definable using first-order logic exists. The object *R* discussed above cannot be constructed in this fashion, and is therefore not a ZFC set. In some extensions of ZFC, objects like *R* are called proper classes. ZFC is silent about types, although some argue that Zermelo's axioms tacitly presupposes a background type theory.

Through the work of Zermelo and others, especially John von Neumann, the structure of what some see as the "natural" objects described by ZFC eventually became clear; they are the elements of the von Neumann universe, *V*, built up from the empty set by transfinitely iterating the power set operation. It is thus now possible again to reason about sets in a non-axiomatic fashion without running afoul of Russell's paradox, namely by reasoning about the elements of *V*. Whether it is *appropriate* to think of sets in this way is a point of contention among the rival points of view on the philosophy of mathematics.

Other resolutions to Russell's paradox, more in the spirit of type theory, include the axiomatic set theories New Foundations and Scott-Potter set theory.

History

Exactly when Russell discovered the paradox is not known. It seems to have been May or June 1901, probably as a result of his work on Cantor's theorem that the number of entities in a certain domain is smaller than the number of subclasses of those entities.[3] He first mentioned the paradox in a 1901 paper in the *International Monthly*, entitled "Recent work in the philosophy of mathematics." He also mentioned Cantor's proof that there is no greatest cardinal, adding that "the master" had been guilty of a subtle fallacy that he would discuss later. Russell also mentioned the paradox in his *Principles of Mathematics* (not to be confused with the later *Principia Mathematica*), calling it "The Contradiction."[4] Again, he said that he was led to it by analyzing Cantor's "no greatest cardinal" proof.

Famously, Russell wrote to Frege about the paradox in June 1902, just as Frege was preparing the second volume of his *Grundgesetze der Arithmetik*.[5] Frege hurriedly wrote an appendix admitting to the paradox, and proposed a solution that was later proved unsatisfactory. In any event, after publishing the second volume of the *Grundgesetze*, Frege wrote little on mathematical logic and the philosophy of mathematics.

Zermelo, while working on the axiomatic set theory he published in 1908, also noticed the paradox but thought it beneath notice, and so never published anything about it.

In 1923, Ludwig Wittgenstein proposed to "dispose" of Russell's paradox as follows:

> "The reason why a function cannot be its own argument is that the sign for a function already contains the prototype of

> its argument, and it cannot contain itself. For let us suppose that the function F(fx) could be its own argument: in that case there would be a proposition 'F(F(fx))', in which the outer function F and the inner function F must have different meanings, since the inner one has the form O(f(x)) and the outer one has the form Y(O(fx)). Only the letter 'F' is common to the two functions, but the letter by itself signifies nothing. This immediately becomes clear if instead of 'F (Fu)' we write '(do) : F(Ou) . Ou = Fu'. That disposes of Russell's paradox." (*Tractatus Logico-Philosophicus*, 3.333)

Russell and Alfred North Whitehead wrote their three-volume *Principia Mathematica* (*PM*) hoping to achieve what Frege had been unable to do. They sought to banish the paradoxes of naive set theory by employing a theory of types they devised for this purpose. While they succeeded in grounding arithmetic in a fashion, it is not at all evident that they did so by purely logical means. While *PM* avoided the known paradoxes and allows the derivation of a great deal of mathematics, its system gave rise to new problems.

In any event, Kurt Gödel in 1930–31 proved that while the logic of much of *PM*, now known as first-order logic, is complete, Peano arithmetic (a fundamental part of any mathematics worth thinking about) is necessarily incomplete if it is consistent. This is very widely – though not universally – regarded as having shown the logicist program of Frege to be impossible to complete.

Applied versions

There are some versions of this paradox that are closer to real-life situations and may be easier to understand for non-logicians. For example, the Barber paradox supposes a barber who shaves men if and only if they do not shave themselves. When one thinks about whether the barber should shave himself or not, the paradox begins to emerge.

As another example, consider five lists of encyclopedia entries within the same encyclopedia:

List of articles about people: ▪ Ptolemy VII of Egypt ▪ Hermann Hesse ▪ Don Nix ▪ Don Knotts ▪ Nikola Tesla ▪ Sherlock Holmes ▪ Emperor Kōnin ▪ Chuck Norris	List of articles starting with the letter L: ▪ L ▪ L!VE TV ▪ L&H ... ▪ List of articles starting with the letter K ▪ List of articles starting with the letter L ▪ List of articles starting with the letter M ...	List of articles about places: ▪ Leivonmäki ▪ Katase River ▪ Enoshima	List of articles about Japan: ▪ Emperor Kōnin ▪ Katase River ▪ Enoshima	List of all lists that do not contain themselves: ▪ List of articles about Japan ▪ List of articles about places ▪ List of articles about people ... ▪ List of articles starting with the letter K ▪ List of articles starting with the letter M ... ▪ **List of all lists that do not contain themselves**?

If the "List of all lists that do not contain themselves" contains itself, then it does not belong to itself and should be removed. However, if it does not list itself, then it should be added to itself.

While appealing, these layman's versions of the paradox share a drawback: an easy refutation of the Barber paradox seems to be that such a barber does not exist. The whole point of Russell's paradox is that the answer "such a set does not exist" means the definition of the notion of set within a given theory is unsatisfactory. Note the difference between the statements "such a set does not exist" and "such a set is empty".

A notable exception to the above may be the Grelling-Nelson paradox, in which words and meaning are the elements of the scenario rather than people and hair-cutting. Though it is easy to refute the Barber's paradox by saying that such a barber does not (and *cannot*) exist, it is impossible to say something similar about a meaningfully defined word.

One way that the paradox has been dramatised is as follows:

Suppose that every public library has to compile a catalog of all its books. The catalog is itself one of the library's books, but while some librarians include it in the catalog for completeness, others leave it out, as being self-evident.

Now imagine that all these catalogs are sent to the national library. Some of them include themselves in their listings, others do not. The national librarian compiles two master catalogs - one of all the catalogs that list themselves, and one of all those which don't.

The question is now, should these catalogs list themselves? The 'Catalog of all catalogs that list themselves' is no problem. If the librarian doesn't include it in its own listing, it is still a true catalog of those catalogs that do include themselves. If he does include it, it remains a true catalog of those that list themselves.

However, just as the librarian cannot go wrong with the first master catalog, he is doomed to fail with the second. When it comes to the 'Catalog of all catalogs that don't list themselves', the librarian cannot include it in its own listing, because then it would belong in the other catalog, that of catalogs that do include themselves. However, if the librarian leaves it out, the catalog is incomplete. Either way, it can never be a true catalog of catalogs that do not list themselves.

Applications and related topics

The Barber paradox, in addition to leading to a tidier set theory, has been used twice more with great success: Kurt Gödel proved his incompleteness theorem by formalizing the paradox, and Turing proved the undecidability of the Halting problem (and with that the Entscheidungsproblem) by using the same trick.

Russell-like paradoxes

As illustrated above for the Barber paradox, Russell's paradox is not hard to extend. Take:

- A transitive verb <V>, that
- can be applied to its substantive form.

Form the sentence:

> The <V>er that <V>s all (and only those) who don't <V> themselves,

Sometimes the "all" is replaced by "all <V>ers".

An example would be "paint":

> The *paint*er that *paint*s all (and only those) that don't *paint* themselves.

or "elect"

> The *elect*or (representative), that *elect*s all that don't *elect* themselves.

Paradoxes that fall in this scheme include:

- The barber with "shave".
- The original Russell's paradox with "contain": The container (Set) that contains all (containers) that don't contain themselves.

- The Grelling-Nelson paradox with "describer": The describer (word) that describes all words, that don't describe themselves.
- Richard's paradox with "denote": The denoter (number) that denotes all denoters (numbers) that don't denote themselves. (In this paradox, all descriptions of numbers get an assigned number. The term "that denotes all denoters (numbers) that don't denote themselves" is here called *Richardian*.)

Related paradoxes

- The liar paradox and Epimenides paradox, whose origins are ancient.
- The Kleene-Rosser paradox, showing that the original lambda calculus is inconsistent, by means of a self-negating statement.
- Curry's paradox (named after Haskell Curry) which does not require negation.
- The smallest uninteresting integer paradox.

See also

- Self-reference
- Universal set

Footnotes

1. ^ Adapted from Potter (2004: 24-25).
2. ^ See Willard Quine, 1938, "On the theory of types," *Journal of Symbolic Logic 3*.
3. ^ In modern terminology, the cardinality of a set is strictly less than that of its power set.
4. ^ Russell, Bertrand (1903). *Principles of Mathematics*. Cambridge: Cambridge University Press, Chapter X, section 100. ISBN 0-393-31404-9.
5. ^ Russell's letter and Frege's reply are translated in Jean van Heijenoort, 1967, and in Frege's *Philosophical and Mathematical Correspondence.*

References

- Potter, Michael, 2004. *Set Theory and its Philosophy*. Oxford Univ. Press.

External links

- Stanford Encyclopedia of Philosophy: "Russell's Paradox" -- by A. D. Irvine.
- Russell's Paradox at cut-the-knot
- Some paradoxes - an anthology